From an iteration formula to Poincaré's Isochronous Center Theorem for holomorphic vector fields

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ABSTRACT. We first generalize a classical iteration formula for one variable holomorphic mappings to a formula for higher dimensional holomorphic mappings. Then, as an application, we give a short and intuitive proof of a classical theorem, due to H. Poincaré, for the condition under which a singularity of a holomorphic vector field is an isochronous center.

1. An iteration formula

Let f be a holomorphic function germ at the origin in the complex plane with $f(0) = 0$. For each positive integer k, we denote by f^k the k-th iteration of f defined as $f^1 = f$, $f^2 = f \circ f$, ... , $f^k = f \circ f^{k-1}$ inductively, which is a well defined holomorphic function germ at the origin.

Assume that $\lambda = f'(0)$ is a primitive m-th root of unity, say, $\lambda^m = 1$ but $\lambda^j \neq 1$ for each positive integer j with $j < m$. Then it is interesting that there exists a positive integer r, such that the m-th iteration f^m has a power series expansion

$$f^m(z) = z + a_1 z^{rm+1} + a_2 z^{rm+2} + \dots$$

at the origin: all the terms of degrees from 2 to rm vanish! This can be proved by applying Rouché's theorem (see [12]). In this section we generalize this formula to germs of higher dimensional mappings by using normal form theory. We denote by $\mathbb{C}^n$ the complex vector space and by Δ^n a ball in $\mathbb{C}^n$ centered at the origin.

PROPOSITION 1 (Iteration Formula). *Let $f : \Delta^n \to \mathbb{C}^n$ be a holomorphic mapping given by*

$$f(z) = \lambda z + o(|z|), \ z \in \Delta^n,$$

where λ is a primitive m-th root of unity. Then, in a neighborhood of the origin,

(1.1) $$f^m(z) = z + o(|z|^m).$$

In the proposition, $z = (z_1, \dots, z_n)$ and the expression $o(|z|^m)$ means that each component of the mapping $f^m(z) - z$ is a power series in $z_1, \dots, z_n$ consisting of terms of degree $> m$.

PROOF. By the hypothesis and a fundamental result in the normal form theory (see [1] or pages 84–85 in [2] for the proof), there exists a biholomorphic transformation in the form of

$$(1.2) \qquad z = (z_1, \ldots, z_n) = h(x_1, \ldots, x_n) = (x_1, \ldots, x_n) + o(|x|)$$

of coordinate in a neighborhood of the origin such that each component g_j of $g = h^{-1} \circ f \circ h = (g_1, \ldots, g_n)$ has a power series expansion

$$(1.3) \qquad g_j(x_1, \ldots, x_n) = \lambda x_j + \sum c^j_{i_1 \ldots i_n} x_1^{i_1} \ldots x_n^{i_n} + o(|x|^m), \quad j = 1, \ldots, n,$$

in a neighborhood of the origin, where $x = (x_1, \ldots, x_n)$ and the sum extends over all n-tuples $(i_1, \ldots, i_n)$ of nonnegative integers with

$$2 \le i_1 + \cdots + i_n \le m \text{ and } \lambda^{i_1 + \cdots + i_n} = \lambda.$$

On the other hand, since λ is a primitive m-th root of unity, we have

$$\lambda^{i_1 + \cdots + i_n} \ne \lambda \text{ if } 2 \le i_1 + \cdots + i_n \le m.$$

Therefore, the sum in the equation (1.3) vanishes, and then we have

$$g(x) = \lambda x + o(|x|^m),$$

and then, considering that $\lambda^m = 1$, we conclude that the m-th iteration g^m can be expressed as

$$(1.4) \qquad g^m(x) = x + o(|x|^m).$$

By (1.2) it is clear that $o(|h^{-1}(z)|^m) = o(|z|^m)$, and then, by (1.4) it is easy to see that

$$\begin{aligned} f^m(z) &= h \circ g^m \circ h^{-1}(z) = h(h^{-1}(z) + o(|h^{-1}(z)|^m)) \\ &= h(h^{-1}(z) + o(|z|^m)) = h(h^{-1}(z)) + o(|z|^m) \\ &= z + o(|z|^m). \end{aligned}$$

This completes the proof. ∎

2. Poincaré's condition for isochronous centers

Consider an n-dimensional complex holomorphic system

$$(2.1) \qquad \dot{z} = F(z), \quad z \in \Delta^n,$$

such that the origin $z = 0$ is a singularity. This means that F is a holomorphic mapping from Δ^n into $\mathbb{C}^n$ such that $F(0) = 0$.

The origin is called a *center* of the system if it has a punctured neighborhood that is filled with periodic orbits, and called an *isochronous* center if it has a punctured neighborhood that is filled with periodic orbits of the same period. Here, and through out this paper, the *period* of a periodic orbit means the smallest positive one.

The problem to find the condition so that a singularity of a system is a center has a long history. The term *center* was defined by H. Poincaré, while the research of center phenomena were started in 1673 when Huygens studied the cycloidal

pendulum (see [7]). As an application of the previous proposition, we shall present a short and intuitive proof of the following classical theorem due to H. Poincaré.

Isochronous Center Theorem. *If the Jacobian matrix $F'(0)$ of F at the origin equals to $\omega i I$ for some real number $\omega \neq 0$, where $i = \sqrt{-1}$ and I is the unit matrix, then the origin is an isochronous center with period $2\pi/|\omega|$.*

This result follows from Poincaré's linearization theorem, which asserts that the system (2.1) is linearizable at the origin via a biholomorphic transformation of coordinates, provided that the n-tuple $(\lambda_1, \ldots, \lambda_n)$ of all eigenvalues of $F'(0)$ is in the Poincaré domain: the convex hull of these eigenvalues in the complex plane does not contain the origin, and that there is no resonance: for any n-tuple $(i_1, \ldots, i_n)$ of nonnegative integers with $i_1 + \cdots + i_n \geq 2$, $\lambda_j \neq \lambda_1^{i_1} \ldots \lambda_n^{i_n}$ for each $j = 1, 2, \ldots, n$ (see Chapter 5 in [1] for the details).

It is interesting that in the history of the study of central singularities, special cases of the Isochronous Center Theorem have been rediscovered several times. For example, when $n = 1$, it was rediscovered by Gregor [9] in 1958, Lukashevich [11] in 1965, Brickman-Thomas [3] in 1977, Villarini [16] in 1992, and Christopher-Devlin [6] in 1997 (see [4], [5], [8], [10], [14] and [15] for other proofs and related topics for the case $n = 1$). In 1998, Needham-McAllister [13] rediscovered the result for two-dimensional systems via the singularity theory of C. H. Briot and J. C. Bouquet. It seems the approach in [13] applies to arbitrary dimensional case.

It is easy to see that a necessary condition so that the origin is an isochronous center of the system (2.1) is that all eigenvalues of $F'(0)$ are pure imaginary with the same absolute value. The converse fails in general.

EXAMPLE 1. *For the system*
$$(\dot{x}, \dot{y}) = (ix, -iy + xy^2), \ (x, y) \in \mathbb{C}^2,$$
it is easy to verify that the corresponding flow is given by
$$\phi(t, (x, y)) = (xe^{it}, \frac{ye^{-it}}{1 - txy}).$$
Clearly, the origin is not a center of the system.

3. Proof of the Isochronous Center Theorem

The following result is well known (see [17] for a simple proof of a more general version, where the singularity is just assumed to be isolated).

LEMMA 1. *If the Jacobian matrix $F'(0)$ of F at the origin is invertible, then there exist a positive number T_0 and a ball B centered at the origin, such that the system (2.1) has no periodic orbit in $B^* = B\backslash\{0\}$ with period less than T_0.*

LEMMA 2. *There is a ball $B \subset \Delta^n$ centered at the origin such that the local flow $\phi(t, z)$ of (2.1) is well defined and real analytic on $[0, 1] \times B$, complex holomorphic with respect to z,*
$$\phi([0, 1] \times B) \subset \Delta^n,$$
and, for each real number $\tau \in [0, 1]$, the Jacobian matrices $\Phi_\tau'(0)$ and $F'(0)$ of the time-τ map Φ_τ and F at the origin, respectively, satisfy
$$\Phi_\tau'(0) = e^{\tau F'(0)}.$$

The previous result is fundamental to the theory of holomorphic vector fields. The time-τ map indicates the mapping $\Phi_\tau : B \to \mathbb{C}^n$ given by

$$\Phi_\tau(z) = \phi(\tau, z), z \in B,$$

and the expression $e^{\tau F'(0)}$ means the matrix $\sum_{k=0}^\infty \frac{(\tau F'(0))^n}{n!}$. If $F'(0) = 2\pi i I$, for example, then $e^{\tau F'(0)} = e^{2\pi i \tau} I$, where I is the unit matrix.

Proof of the Isochronous Center Theorem. Without loss of generality, assume $F'(0) = 2\pi i I$. We shall show that the origin is an isochronous center with period 1.

Let B be a ball centered at the origin that is determined by Lemma 2. Let $\Phi_1 : B \to \mathbb{C}^n$ be the time-1 map of the local flow $\phi(t, z)$ of the system (2.1). Then by Lemma 2 we have $\Phi_1'(0) = I$, which implies that the equation

(3.1) $\Phi_1(z) = z + o(|z|^m), \ z \in B,$

holds for $m = 1$. But, we can show that this equation holds for all integers $m \geq 1$!

For any positive integer $m > 1$, consider the time-$\frac{1}{m}$ map $\Phi_{\frac{1}{m}}$ of the local flow ϕ. By Lemma 2, we have $\Phi_{\frac{1}{m}}'(0) = e^{\frac{2\pi i}{m}} I$, and then by the proposition, the m-th iteration $\Phi_{\frac{1}{m}}^m$ of $\Phi_{\frac{1}{m}}$ can be expressed as

$$\Phi_{\frac{1}{m}}^m(z) = z + o(|z|^m), \ z \in B.$$

Thus (3.1) holds for all positive integers m. For

$$\Phi_{\frac{1}{m}}^m(z) = \phi(\frac{m}{m}, z) = \phi(1, z) = \Phi_1(z), \ z \in B.$$

Since Φ_1 is a holomorphic mapping on B, we have proved that $\Phi_1 = id_B$, the identity mapping on B. Therefore, all orbits of the system that intersect $B^* = B \backslash \{0\}$ are periodic, and then the origin is a center.

Now, let us show that the origin is, in fact, an isochronous center with period 1. We first show that for each $k \geq 2$, the origin is an isolated fixed point of the time-$\frac{1}{k}$ map $\Phi_{\frac{1}{k}}$, of the local flow. Otherwise, by the inverse function theorem, the Jacobian determinant of the mapping $z \mapsto \Phi_{\frac{1}{k}}(z) - z$ vanishes at $z = 0$, and then $\Phi_{\frac{1}{k}}'(0)$ must have an eigenvalue equal to 1. But by Lemma 2, $\Phi_{\frac{1}{k}}'(0) = e^{\frac{2\pi i}{k}} I$, and then we have $k = 1$. Contradiction! Thus the origin is an isolated fixed point of $\Phi_{\frac{1}{k}}$. Therefore, for an arbitrarily given integer $k_0 \geq 2$, there exists a neighborhood of the origin in which the system has no periodic orbit of periods $\frac{1}{2}, \ldots, \frac{1}{k_0}$.

On the other hand, by the equation $\Phi_1 = id_B$, it is clear that the period of any periodic orbit of the system that intersects B^* must divide 1, and then it equals to $\frac{1}{k}$ for some positive integer k.

Thus, there exists a punctured neighborhood of the origin in which each periodic orbit of the system either has period 1, or has period less than $\frac{1}{k_0}$. Hence, by Lemma 1 and the arbitrariness of k_0, we conclude that there exists a puncture neighborhood of the origin in which all periodic orbits of the system has period 1. This completes the proof. ∎

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